Properties of Khovanov homology for positive braid knots

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Abstract

In this paper we solve one open problem from [13] and give some generalizations. Namely, we prove that the first homology group of positive braid knot is trivial. Also, we show that the same is true for the Khovanov-Rozansky homology [15] ($sl(n)$ link homology) for any positive integer n.

1 Introduction

In recent years there has been a lot of interest in the "categorification" of link invariants, initiated by Khovanov in [10]. For each link L in S^3 he defined a graded chain complex, with grading preserving differentials, whose graded Euler characteristic is equal to the Jones polynomial of the link L ([8],[9]). This is done by starting from the state sum expression for the Jones polynomial (which is written as an alternating sum), then constructing for each term a module whose graded dimension is equal to the value of that term, and finally, constructing the differentials as appropriate grading preserving maps, so that the complex obtained is a link invariant. There is also similar construction [15], for the categorification of the n-specializations of

HOMFLYPT polynomial ([5], [21], [18]) as well as the categorifications of various link and graph polynomial invariants ([12], [11], [2], [7]).

Although the theory is rather new, it already has strong applications. For instance, the proof of Milnor conjecture by Rasmussen in [19] and the existence of the exotic differential structure on $\mathbb{R}^4$ ([20],[6] page 522), which were previously accesible only by huge machinery of gauge theory.

The advantage of Khovanov homology theory is that its definition is combinatorial and since there is a straightforward algorithm for computing it, it is (theoretically) highly calculable. Nowadays there are several computer programs [3], [22] that can calculate effectively Khovanov homology of links with up to 50 crossings.

Based on the calculations there are many conjectures about the properties of link homology see e.g. [1], [13], [4]. Some of the properties have been proved till now (see [16], [17]), but many of them are still open.

In this paper we prove the conjecture from [13] that the first homology group of the positive braid knot is trivial. In the proof we use the basic ingredients of the construction of homology (cubic complex, independence of the planar projection chosen), and so major part can be directly applied to other link homology theories. Especially, in the case of $sl(n)$-link homology [15], whose particular definitions of chain groups and differentials make it practically incalculable, we modify slightly our proof to show that the first $sl(n)$ homology group of positive braid knot is trivial, as well.

Since torus knots are positive braid knots, we automatically obtain the same properties for them. Even more, in the sequel papers [23], [24], we obtain further properties of $sl(2)$ and $sl(n)$-link homology for torus knots.

The organization of the paper is the following: in Section 2 we briefly recall the definition of the Khovanov homology. In Section 3 we give the main result. Finally, in Section 4 we adapt the proof of the main result for $sl(n)$ case.
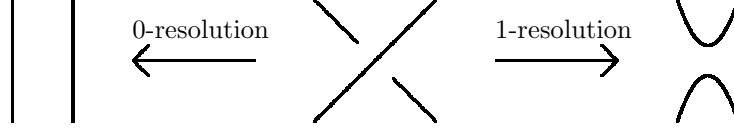
2   Notation

We recall briefly the definition of Khovanov homology for links. For more details see [1],[10].

First of all, take a link K, its planar projection D, and take an ordering of the crossings of D. For each crossing c of D, we define 0-resolution and

1-resolution, according to the following picture:

Denote by m the number of crossings of D. Then there is bijective correspondence between the total resolutions of D and the set $\{0,1\}^m$. Namely, to every m-tuple $\epsilon = (\epsilon_1, \ldots, \epsilon_m) \in \{0,1\}^m$ we associate the resolution D_ϵ where we resolved the i-th crossing in a ϵ_i-resolution.

Every resolution D_ϵ is a collection of disjoint circles. To each circle we associate graded $\mathbb{Z}$-module V, which is freely generated by two basis vectors 1 and X, with $\deg 1 = 1$ and $\deg X = -1$. To D_ϵ we associate the module M_ϵ, which is the tensor product of V's over all circles in the resolution. We group all the resolution D_ϵ with fixed $|\epsilon|$ (sum of elements of ϵ). We draw all resolutions as (skewed) m-dimensional cube such that in i-th column are the resolutions D_ϵ with $|\epsilon| = i$. We define the i-th chain group C^i by:

$$C^i(D) = \oplus_{|\epsilon|=i} M_\epsilon\{i\}.$$

Here, by $\{i\}$, we have denoted shift in grading of M_ϵ (for more details see e.g. [1]).

The differential $d^i : C^i(G) \to C^{i+1}(G)$ is defined as (signed) sum of "per-edge" differentials. Namely, the only nonzero maps are from D_ϵ to $D_{\epsilon'}$, where $\epsilon = (\epsilon_1, \ldots, \epsilon_m)$, $\epsilon_i \in \{0,1\}$, if and only if ϵ' has all entries same as ϵ except one ϵ_j, for some $j \in \{0,1\}$, which is changed from 0 to 1. We denote these differentials by d_ν, where ν is m-tuple which consists of the label $*$ at the position j and of $m-1$ 0's and 1's (the same as the remaining entries of ϵ). Note that in these cases, either two circles of D_ϵ merge into one circle of $D_{\epsilon'}$ or one circle of D_ϵ splits into two circles of $D_{\epsilon'}$, and all other circles remain the same.

In the first case, we define the map d_ν as the identity on the tensor factors (V) that correspond to the unchanged circles, and on the remaining factors we define as the (graded preserving) multiplication map $m : V \otimes V \to V\{1\}$, which is given on basis vectors by:

$$m(1 \otimes 1) = 1, \quad m(1 \otimes X) = m(X \otimes 1) = X, \quad m(X \otimes X) = 0.$$

In the second case, we define the map d_ν as the identity on the tensor factors (V) that correspond to the unchanged circles, and on the remaining

factors we define as the (graded preserving) comultiplication map $\Delta : V \to V \otimes V\{1\}$, which is given on basis vectors by:

$$\Delta(1) = 1 \otimes X + X \otimes 1, \quad \Delta(X) = X \otimes X.$$

Finally, to obtain the differential d^i of the chain complex $C(D)$, we sum all contributions d_ν with $|\nu| = i$, multiplied by the sign $(-1)^{f(\nu)}$, where $f(\nu)$ is equal to the number of 1's ordered before $*$ in ν. This makes every square of our cubic complex anticommutative, we obtain the genuine differential (i.e. $(d^i)^2 = 0$).

The homology groups of the obtained complex $(C(D), d)$ we denote by $H^i(D)$ and call *unnormalized homology groups of* D. In order to obtain link invariants (i.e. independence of the chosen projection), we have to shift the chain complex (and hence the homology groups) by:

$$\mathcal{C}(D) = C(D)[-n_-]\{n_+ - 2n_-\}, \tag{1}$$

where n_+ and n_- are number of positive and negative crossings, respectively, of the diagram D.

positive   negative

In the formula (1), we have denoted by $[-n_-]$, the shift in homology degrees (again, for more details see [1]).

The homology groups of the complex $\mathcal{C}(D)$ we denote by $\mathcal{H}^i(D)$. Hence, we have $\mathcal{H}^{i,j}(D) = H^{i+n_-, j-n_+ +2n_-}(D)$.

Theorem 1 *([10],[1]) The homology groups $\mathcal{H}(D)$ are independent of the choice of the planar projection D. Furthermore, the graded Euler characteristic of the complex $\mathcal{C}(D)$ is equal to Jones polynomial of the link K.*

Hence, we can write $\mathcal{H}(K)$, and we call $\mathcal{H}^i(K)$ *homology groups of link* K.

Also if the diagram D has only positive crossings then we do not have shift in the homology degrees, and so we have that, for example, $\mathcal{H}^1(K)$ is trivial if and only if $H^1(D)$ is trivial. Also, in general case if we have a positive knot K (the knot that has a planar projection with only positive

4

crossings) then we have that $\mathcal{H}^i(K)$ is trivial for all $i < 0$. Further, if D is planar projection of positive knot K with n_- negative crossings then $H^i(D)$ is trivial for $i < n_-$.

3   Positive braid knots

The positive braid knots are the knots (or links) that are the closures of positive braids. Let K be arbitrary positive braid knot and let D be its planar projection which is the closure of a positive braid. Denote the number of strands of that braid by p. We say that the crossing c of D is of the type σ_i, $i < p$, if it corresponds to the generator σ_i in the braid word of which D is the closure.
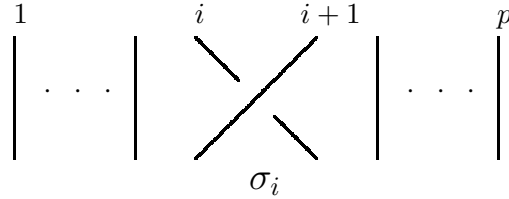
Denote the number of crossings of the type σ_i by l_i, $i = 1, \ldots, p-1$ and order them from top to bottom. Then each crossing c of D we can write as the pair (i, α) (we will also write $(i\alpha)$ if there is no possibility of confusion), $i = 1, \ldots, p-1$ and $\alpha = 1, \ldots, l_i$, if c is of the type σ_i and it is ordered as α-th among the crossings of the type σ_i. Finally, we order the crossings of D by the following ordering: $c = (i\alpha) < d = (j\beta)$ if and only if $i < j$, or $i = j$ and $\alpha < \beta$.

Since the positive braid knot is positive knot, we have that $\mathcal{H}^i(K) = 0$ for $i < 0$. Also we know its zeroth homology group (see e.g. [13]) is two-dimensional (without torsion) and that the q-gradings of the two generators are $1 - p + n(D) \pm 1$, where $n(D)$ is the number of crossings of D. In the following theorem we prove that the first homology group of the positive braid knot is trivial.

**Theorem 2** *If K is positive braid knot, than $\mathcal{H}^1(K) = 0$.*

Proof:

First of all, if D is the regular diagram of K, which is the closure of a positive braid, then we have that $\mathcal{H}^{i,j}(K) = H^{i,j-n(D)}(D)$, where $n(D)$ is the number of crossings of D, and so we have that $\mathcal{H}^1(K) = 0$ if and only if

$H^1(D) = 0$. So we are going to show that the (unnormalized) first homology group of D is trivial.

In order to prove that $H^1(D) = 0$, we will use the definition of Khovanov homology, i.e. we will prove that for any element t' from chain group $C^1(D)$ such that $d^1(t') = 0$ there exists an element $t \in C^0(D)$ such that $t' = d^0(t)$. For this we first need to understand the chain groups $C^0(D)$, $C^1(D)$ and $C^2(D)$ and the differentials d^0 and d^1.

$C^0(D)$ "comes" from all resolutions K_s with $|s| = 0$. However, since D is the closure of the positive braid we have only one such resolution s_0 (all crossings are resolved into 0-resolutions) and it is an unlink that consists of p unknots (i.e. the closure of the trivial braid with p strands). Hence, we have that $C^0(D) = V^{\otimes p}$ where we assigned the i-th copy of V (denoted by V^i) to the circle which is the closure of the i-th strand of D_{s_0}.

Now, we pass to $C^1(D)$. It "comes" from all resolutions D_s with $|s| = 1$, i.e. all the resolutions were we resolve all except one crossing of D in a 0-resolutions and the remaining one in 1-resolution. In such way, if the crossing c that is resolved into 1-resolution is of the type σ_i (i.e. if $c = (i\alpha)$, for some $\alpha = 1, \ldots, l_i$), then the corresponding resolution, D_c, is the closure of the plat diagram E_i (see picture).
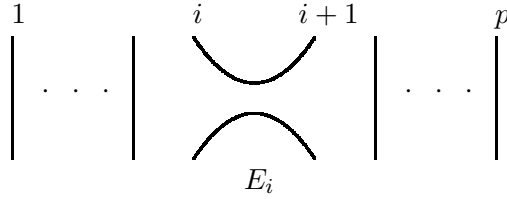
To that resolution we assign the vector space $V_c = V^{\otimes (p-1)}$, where we have assigned the first $i-1$ and the last $p-1-i$ copies of V to the circles that are the closures of the first $i - 1$ and the last $p - i$ strands of the resolution D_c, respectively, and the i-th copy of V corresponds to the remaining circle (closure of E_i). So, we have that $C^1(D) = \bigoplus_{c \in c(D)} V_c\{1\}$. Further on, we denote the k-th copy of V in V_c by V_c^k.

The differential $d^0 : C^0(D) \to C^1(D)$ is given by the maps $f_c : V^{\otimes p} \to V_c$ where if c is of the type σ_i than f_c acts as the identity on the first $i - 1$ and the last $p-i-1$ copies of V and as the multiplication m on the remaining two copies of V. In other words, f_c maps the copies V^j as the identity onto V_c^j, for $j < i$, maps the copies V^{j+1} as the identity onto V_c^j, for $i < j < p$, and on the remaining two factors act as the multiplication $m : V^i \otimes V^{i+1} \to V^i$.

Further, $C^2(D)$ comes from the resolutions where exactly two of the crossings are resolved in 1-resolutions and the remaining ones are resolved in 0-rezolutions. Denote the two crossings that are resolved in 1-resolution by c and d (where c is ordered before d), and let c be of σ_i type and d of σ_j type. Then we have that $i \leq j$.

If $i = j$ then the corresponding resolution $D_{c,d}$ is a closure of a plat diagram E_i^2, has p circles and hence the corresponding summand $V_{c,d}$ of a $C^2(D)$ is isomorphic to $V^{\otimes p}$. Here we have assigned the first $i - 1$ and the last $p - i - 1$ copies of V to the circles that are the closures of the first $i - 1$ and the last $p - i - 1$ strands of the resolution $D_{c,d}$, respectively, while i-th and $(i + 1)$-th copy of V correspond to the remaining two circle that are formed of i-th and $(i + 1)$-th strand (i-th copy of V to the outer, and $(i + 1)$-th copy of V to the inner circle).

If $i < j$ then the corresponding resolution is a closure of a plat diagram $E_i E_j$ (or $E_j E_i$), has $p - 2$ circles and hence the corresponding summand $V_{c,d}$ of a $C^2(D)$ is isomorphic to $V^{\otimes(p-2)}$.

If $i + 1 < j$, we assign the first $i - 1$ copies of V to the closures of the first $i - 1$ strands. The i-th copy of V is assigned to the circle that is obtained by joining the i-th and $(i + 1)$-th strand (the 1-resolution of c). The following $j - i - 2$ copies of V are assigned to the closures of the strands from $(i+2)$-th to $(j - 1)$-th of $D_{c,d}$, respectively. The $(j - 1)$-th copy of V is assigned to the circle that is obtained by joining the j-th and $(j + 1)$-th strand (the 1-resolution of d). The remaining $p - j - 1$ copies of V are assigned to the closures of the last $p - j - 1$ strands of $D_{c,d}$.

If $i + 1 = j$, then we assign the first $i - 1$ copies of V to the closures of the first $i - 1$ strands. The i-th copy of V is assigned to the circle that is obtained by joining the i-th, $(i+1)$-th and $(i+2)$-th strand (the 1-resolutions of c and d). The remaining $p - i - 2$ copies of V are assigned to the closures of the strands from $(i + 3)$-th to p-th of $D_{c,d}$, respectively.

In all previous cases, we denote the k-th copy of V in $V_{c,d}$ by $V_{c,d}^k$.

Finally, the second chain group is

$$C^2(D) = \bigoplus_{c,d \in c(D),\, c < d} V_{c,d}\{2\}.$$

Now, we can describe the differential $d^1 : C^1(D) \to C^2(D)$. It is given as a sum of the maps of the form $f_{ecd} : V_e \to V_{c,d}$ (with $c, d, e \in c(D)$, $c < d$), where f_{ecd} is zero unless $e = c$ or $e = d$. Let $c = (i\alpha)$ and $d = (j\beta)$, for some

$1 \leq i \leq j \leq n-1$, $\alpha = 1, \ldots, l_i$ and $\beta = 1, \ldots, l_j$. The maps $f_{ccd} : V_c \to V_{c,d}$ are as follows:

If $i = j$ then f_{ccd} is given by the identity maps: $id : V_c^l \to V_{c,d}^l$, for $l < i$, and $id : V_c^l \to V_{c,d}^{l+1}$, for $i < l < p$, and by comultiplication $\Delta : V_c^i \to V_{c,d}^i \otimes V_{c,d}^{i+1}$.

If $i < j$, then f_{ccd} is given by the identity maps $id : V_c^l \to V_{c,d}^l$, for $l < j - 1$, and $id : V_c^{l+1} \to V_{c,d}^l$, for $j < l < p - 1$, and by multiplication $m : V_c^{j-1} \otimes V_c^j \to V_{c,d}^{j-1}$.

The other class of nonzero maps $f_{dcd} : V_d \to V_{c,d}$ is in the case $i = j$ given by $-f_{ccd}$ (since in this case $V_c = V_d$, and $c < d$), and in the case $i < j$ is given as $-g_{cd}$, where g_{cd} is given by the identity maps: $id : V_c^l \to V_{c,d}^l$, for $l < i$, and $id : V_c^{l+1} \to V_{c,d}^l$, for $i < l < p - 1$, and by multiplication $m : V_c^i \otimes V_c^{i+1} \to V_{c,d}^i$.

Now, we can go back to the proof. Let

$$t' = (t_{1,1}, \ldots, t_{1,l_1}, t_{2,1}, \ldots, t_{2,l_2}, \ldots, t_{p-1,1}, \ldots, t_{p-1,l_{p-1}}) \in C^1(D)$$

be such that $d^1(t') = 0$. Here we have that $t_{i,\alpha} \in V_{(i\alpha)}$ for $i = 1, \ldots, p-1$, $\alpha = 1, \ldots, l_i$. Our aim is to find an element $t \in C^0 = V^{\otimes p}$ such that $d^0(t) = t'$.

Since $d^1(t') = 0$ we have that its projection, denoted by $d^1_{i\alpha\beta}$, to the space $V_{i\alpha, i\beta}$ is equal to zero, for every $i = 1, \ldots, p-1$, and $\alpha, \beta = 1, \ldots, l_i$. However, this implies that $t_{i\alpha} = t_{i\beta}$ for every $i = 1, \ldots, p-1$, $\alpha, \beta = 1, \ldots, l_i$, since only the maps from $V_{(i\alpha)}$ and $V_{(i\beta)}$ to $V_{(i\alpha),(i\beta)}$ are nonzero, and the only nonidentity part of the mappings is the comultiplication Δ on the same (i-th) copy of V in both $V_{i\alpha}$ and $V_{i\beta}$.

Hence, we have obtained that $t' \in \ker d^1$ if and only if $t_{i\alpha} = t_{i\beta}$ for every $i = 1, \ldots, p-1$, $\alpha, \beta = 1, \ldots, l_i$ and $\bar{t} = (t_{1,1}, t_{2,1}, \ldots, t_{p-1,1}) \in \ker \bar{d}^1$, where $\bar{d}^1$ is the restriction of d^1 to $W = V_{(1,1)} \oplus V_{(2,1)} \oplus \cdots V_{(p-1,1)}$.

Further, note that for every $i = 1, \ldots, p-1$, and $\alpha, \beta = 1, \ldots, l_i$, the projection of $d^0(t)$, for any $t \in C^0(D)$ to $V_{(i\alpha)}$ and $V_{(i\beta)}$ is equal, i.e. the differential d^0 is completely determined by the map $\bar{d}^0$ which is the projection of d^0 on W. Finally, we have that if there exists $t \in C^0$ such $\bar{d}^0(t) = \bar{t}$, then $d^0(t) = t'$. Hence, to finish the proof, we are left with proving that for every $y \in \ker \bar{d}^1 \in W$, there exists $x \in C^0(D)$, such that $\bar{d}^0(x) = y$.

Now observe the positive braid knot K' which has the regular diagram D' which is the closure of the braid $\sigma_1 \sigma_2 \cdots \sigma_{p-1}$ (we omit the letters σ_i's

which are not contained in the braid word of which D is the closure). Its zeroth chain group $\bar{C}^0(D')$ is obviously equal to $C^0(D) = V^{\otimes p}$, the first chain group $\bar{C}^1(D')$ is equal to W and its second chain group $\bar{C}^2(D)$ is equal to $\bigoplus_{1 \leq i < j \leq p-1} V_{(i1),(j1)}$. Its zeroth differential is equal to the previously defined $\bar{d}^0$, while the first differential is equal to $\bar{d}^1$. Since K' is isotopic to the unknot (or to the unlink consisting of the unknots), its first homology group is trivial and hence for every $y \in \ker \bar{d}^1$ there exists $x \in \bar{C}^0(D') = C^0(D)$ such that $\bar{d}^0(x) = y$. This concludes the proof. ∎

4 $sl(n)$ case

In this section we will adapt our proof from the previous section to show that the first Khovanov-Rozansky homology group of a positive braid knot is trivial. We will not define Khovanov-Rozansky $sl(n)$-link homology, for details see [15]. For our purposes it is enough to point out the similarities and the differences between $sl(n)$-link homology and the standard Khovanov ($sl(2)$) homology from Section 2.

The basic principle of the construction, namely the cubic complex, is the same for $sl(n)$ case as it is in the standard Khovanov homology. The difference is that we take the oriented knot and the two (oriented) resolutions of the crossings as given in the following picture:
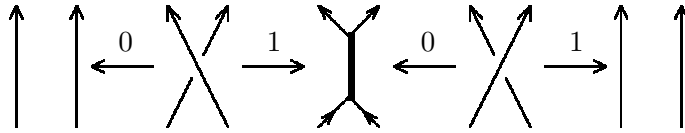
Hence, in this case the (complete) resolution of the planar projection D consists of the three-valent graphs such that at each vertex we have exactly one (unoriented) wide edge and two oriented thin edges, and such that at one end of each wide edge two thin edges are incoming and at the other end are outgoing. Since we will work only with the diagrams that are closure of a braid, we will denote the singular resolution (one with the thick edge) by $\bar{E}_i$ if it comes from the crossing of the type σ_i, and the obtained total resolution of D will be the closure of the sequence of $\bar{E}_i$'s.
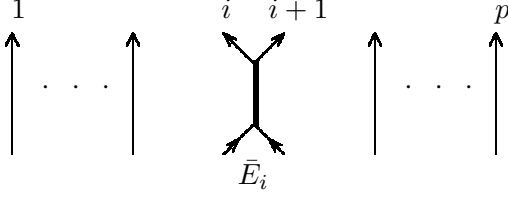
Now, to each such resolution D_ϵ is assigned a graded vector space $\bar{V}_\epsilon\{-|\epsilon|\}$ (the theory is defined over $\mathbb{Q}$), for details see [15], and then after summing along the columns of the cubic complex we obtain the chain groups $C_n^i(D)$ by:

$$C_n^i(D) = \bigoplus_{|\epsilon|=i} \bar{V}_\epsilon.$$

Further, the differential d_n is obtained as a signed sum of "per-edge" maps, and thus we obtain the chain complex $(C_n(D), d_n)$. Denote its homology groups by $H_n^{i,j}(D)$. Finally, after overall shift we obtain the chain complex $\bar{C}_n(D)$ given by

$$\bar{C}_n(D) = C_n(D)[-n_-(D)]\{-(n-1) \cdot n_+(D) + n \cdot n_-(D)\}.$$

Remark 3 *There is also slight difference in q-gradings of the $sl(n)$ and standard $sl(2)$ theory. Namely, if we put $n = 2$ in $sl(n)$ theory ([15]) we obtain standard Khovanov homology ([10]) with q-grading inverted, i.e. the q-gradings of [15] are the negative of the q-gradings of [10]. We would obtain the same convention for the standard Khovanov homology (Section 2) if we define $\deg 1 = -1$, $\deg X = 1$ and all later shifts in q-gradings $\{i\}$ we replace by $\{-i\}$. These conventions for $sl(2)$ theory are used in [14] and [13].*

Denote the homology groups of the complex $\bar{C}_n(D)$ by $\mathcal{H}_n^{i,j}(D)$. Then we have:

Theorem 4 *([15]) The homology groups $\mathcal{H}_n^{i,j}(D)$ are independent of the planar projection D. Even more, the graded Euler characteristic of the complex $\mathcal{C}_n^{i,j}(D)$ is equal to n-specialization of HOMFLYPT polynomial.*

Hence we can write $\mathcal{H}_n^{i,j}(K)$.

Theorem 5 *For every positive braid knot K, we have that the homology group $\mathcal{H}_n^1(K)$ is trivial.*

Proof:

Mainly we will adapt our proof from the previous Section for this case. First of all, we again take the diagram D of knot K, which is the closure of positive braid σ. Since D has only positive crossings we have that $\mathcal{H}^1(K)$ is trivial if and only if $H^1(D)$ is trivial. So, we have to prove that the latter group is trivial.

Again, we use the direct sum definition of the chain groups $\bar{C}_n^0(D)$, $\bar{C}_n^1(D)$, $\bar{C}_n^2(D)$, and of the differentials d_n^0 and d_n^1. We will prove that for each $t' \in \bar{C}_n^1(D)$ such that $d_n^1(t') = 0$, there exists $t \in \bar{C}_n^0(D)$ such that $d_n^0(t) = t'$.

We denote the restriction of t' to the space $\bar{V}_{i\alpha}$ by $t'_{i\alpha}$. Again the differential d_n^1 maps only the spaces $\bar{V}_{i\alpha}$ and $\bar{V}_{i\beta}$ to the $\bar{V}_{i\alpha,i\beta}$, and since $d_n^1(t') = 0$ we have that its restriction to the $\bar{V}_{i\alpha,i\beta}$ is zero.

From the definition we have that $\bar{V}_{i\alpha} = \bar{V}(\bar{E}_i)\{-1\}$ and $\bar{V}_{i\alpha,i\beta} = \bar{V}(\bar{E}_i^2)\{-2\}$. From [15] we have that the last vector space is isomorphic to $\bar{V}(\bar{E}_i)\{-1\} \oplus \bar{V}(\bar{E}_i)\{-3\}$, and the projection of the per-edge map from $\bar{V}_{i\alpha}$ to $\bar{V}_{i\alpha,i\beta}$ onto the first summand of the latter space is identity map.

Hence, we have that $t'_{i\alpha} = t'_{i\beta}$ for every i, α, β, and our problem reduces, like in $sl(2)$ case, to the problem when there is at most one crossing of the type σ_i for every i. In other words we are left with proving that $H_n^1(D')$ is trivial. However, since D' is isotopic to the unknot the triviality of $H_n^1(D')$ follows from the independence of the $sl(n)$ homology of knot projection chosen, which concludes our proof. ∎

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